

On the Axiomatic Systems of Steenrod Homology Theory of Compact Spaces

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Abstract

On the category of compact metric spaces an exact homology theory was defined and its relation to the Vietoris homology theory was studied by Steenrod [S]. In particular, the homomorphism from the Steenrod homology groups to the Vietoris homology groups was defined and it was shown that the kernel of the given homomorphism are homological groups, which was called weak homology groups [S], [E]. The Steenrod homology theory on the category of compact metric pairs was axiomatically described by J.Milnor. In [Mil] the uniqueness theorem is proved using the Eilenberg-Steenrod axioms and as well as relative homeomorphism and cluster axioms. J. Milnor constructed the homology theory on the category Top_C^2 of compact Hausdorff pairs and proved that on the given category it satisfies nine axioms - the Eilenberg-Steenrod, relative homeomorphism and cluster axioms (see theorem 5 in [Mil]). Besides, using the construction of weak homology theory, J.Milnor proved that constructed homology theory satisfies partial continuity property on the subcategory Top_{CM}^2 (see theorem 4 in [Mil]) and the universal coefficient

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formula on the category Top_C^2 (see Lemma 5 in [Mil]). On the category of compact Hausdorff pairs, different axiomatic systems were proposed by N. Berikashvili [B1], [B2], H. Inasaridze and L. Mdzinarishvili [IM], L. Mdzinarishvili [M] and H. Inasaridze [I], but there was not studied any connection between them. The paper studies this very problem. In particular, in the paper it is proved that any homology theory in Inasaridze sense is the homology theory in the Berikashvili sense, which itself is the homology theory in the Mdzinarishvili sense. On the other hand, it is shown that if a homology theory in the Mdzinarishvili sense is exact functor of the second argument, then it is the homology in the Inasaridze sense.

1 Introduction

Let Top_C^2 be the category of compact Hausdorff pairs and continuous maps and Ab be the category of abelian groups.

A sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in \mathbb{Z}}$ of covariant functors $\bar{H}_n : Top_C^2 \rightarrow Ab$ is called homological [M], [ES], if:

1_H) for each object $(X, A) \in Top_C^2$ and $n \in \mathbb{Z}$ there exists a ∂ -homomorphism

$$(1.1) \quad \partial : \bar{H}_n(X, A) \rightarrow \bar{H}_{n-1}(A)$$

($\bar{H}_n(A) \equiv \bar{H}_{n-1}(A, \emptyset)$, where \emptyset is the empty set);

2_H) the diagram

$$(1.2) \quad \begin{array}{ccc} \bar{H}_n(X, A) & \rightarrow & \bar{H}_{n-1}(A; G) \\ \downarrow f_* & & \downarrow (f|_A)_* \\ \bar{H}_n(Y, B) & \rightarrow & \bar{H}_{n-1}(B; G) \end{array}.$$

is commutative for each continuous mapping $f : (X, A) \rightarrow (Y, B)$ ($f_* : \bar{H}_n(X, A) \rightarrow \bar{H}_n(Y, B)$ and $(f|_A)_* : \bar{H}_n(A) \rightarrow \bar{H}_n(B)$ are the homomorphisms induced by $f : (X, A) \rightarrow (Y, B)$ and $f|_A : A \rightarrow B$, correspondingly).

Let \mathcal{Pol}^2 be the full subcategory of the category Top_C^2 , consisting of compact polyhedral pairs.

A homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in \mathbb{Z}}$ defined on the category Top_C^2 is called homology theory in the Eilenberg-Steenrod sense if it satisfies homotopy, excision, exactness and dimension axioms [ES]. It is known that up to an isomorphism such a homology theory is unique on the subcategory \mathcal{Pol}^2 of compact polyhedral pairs [ES] and it is denoted by $H_* = \{H_n\}_{n \in \mathbb{Z}}$, but it is not unique on the category Top_C^2 of compact Hausdorff pairs.

The Steendor homology theory on the category of compact metric pairs was first axiomatically described by J. Milnor [Mil]. He proved the uniqueness theorem using the Eilenberg-Steenrod axioms and additionally two more - relative homeomorphism and clusters axioms:

RH (relative homeomorphism axiom): if $f : (X, A) \rightarrow (Y, B)$ is a map in Top_{CM}^2 which carries $X - A$ homeomorphically onto $Y - B$, then

$$(1.3) \quad f_* : \bar{H}_n(X, A) \rightarrow \bar{H}_n(Y, B)$$

is an isomorphism.

CL(cluster axiom): if X is the union of countable many compact subsets X_1, X_2, \dots which intersect pairwise at a single point $*$, and which have diameters tending to zero, then $\bar{H}_n(X, *)$ is naturally isomorphic to the direct product of the groups $\bar{H}_n(X_i, *)$.

In [Mil] the following is proved:

Theorem 1.1. *(see theorem 3 in [Mil]) Given two homology theories \bar{H}_*^M and \bar{H}_* on the category Top_{CM}^2 , both satisfying the nine axioms (the Eilenberg-Steenrod, relative homeomorphism and cluster axioms), any coefficient isomorphism $\bar{H}_0^M(*) \approx \bar{H}_0(*) \approx G$ extends uniquely to an equivalence between the two homology theories.*

In [Mil] J. Milnor constructed the homology theory \bar{H}_*^M on the category Top_C^2 of compact Hausdorff pairs and gave its several properties. In particular [Mil] the following is proved:

Theorem 1.2. *(see theorem 5 in [Mil]) The homology theory \bar{H}_*^M , defined on the category Top_C^2 of compact Hausdorff pairs, satisfies the nine axioms (the Eilenberg-Steenrod axioms as well as relative homeomorphism and clusre axioms).*

Theorem 1.3. *(see theorem 4 in [Mil]) Let \bar{H}_*^M be a homology theory satisfying the nine axioms (the Eilenberg-Steenrod axioms as well as relative homeomorphism and cluster axioms), and let $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$ be an inverse system of compact metric spaces with inverse limit X . Then there is an exact sequence*

$$(1.4) \quad 0 \rightarrow \varprojlim^1 \bar{H}_{n+1}^M(X_i) \rightarrow \bar{H}_n^M(X) \rightarrow \varprojlim \bar{H}_n^M(X_i) \rightarrow 0$$

for each integer n . A corresponding assertion holds if each space is replaced by a pair.

Theorem 1.4. *(see lemma 5 in [Mil]) The homology theory \bar{H}_*^M is related to the Čech cohomology theory by a split exact sequence*

$$(1.5) \quad 0 \rightarrow Ext(\check{H}^{n+1}(X, A); G) \rightarrow \bar{H}_n^M(X, A; G) \rightarrow Hom(\check{H}^n(X, A); G) \rightarrow 0.$$

As we see the uniqueness theorem was proved only on the category Top_{CM}^2 of compact metric pairs [Mil] and therefore, the problem was open for the category Top_C^2 .

The axiomatic description of the Steenrod homology theory on the category Top_C^2 of compact Hausdorff pairs was given by Berikashvili [B1], [B2]. In particular, in [B1] it is proved that if a homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in \mathbb{Z}}$ defined on the category Top_C^2 of compact Hausdorff pairs satisfies the Eilenberg-Steenrod axioms and the following A, B and C axioms, then it is naturally isomorphic to the Chogoshvili homology theory:

A: The projection $(X, A) \rightarrow (X/A, *)$ induces an isomorphism $\bar{H}_n(X, A) \approx \bar{H}_*(X/A, *)$.

B: For the inverse spectrum of pairs $\{(S_\alpha^n, *), \pi_{\alpha\beta}\}$, where S_α^n is a finite cluster of n -dimension spheres and $\pi_{\alpha\beta}$ maps each sphere of the cluster either to the fixed point or homeomorphically to a cluster sphere, there holds the equality

$$(1.6) \quad \bar{H}_*(\varprojlim \{(S_\alpha^n, *), \pi_{\beta}^\alpha\}) \approx \varprojlim \{\bar{H}_*(S_\alpha^n, *), (\pi_{\alpha\beta})_*\}, \quad n \in \mathbb{Z}.$$

C: The natural homomorphism

$$(1.7) \quad \varinjlim \bar{H}_n(|\mathcal{N}(X)|_p) \rightarrow \bar{H}_n(X), \quad n \in \mathbb{Z},$$

induced by the mapping $\omega : |\mathcal{N}(X)| \rightarrow X$, where $|\mathcal{N}(X)|$ is the limit of the inverse spectrum of realizations of complexes of the spectrum $\mathcal{N}(X) = \{\mathcal{N}_\lambda(X), \pi_{\lambda\mu}\}$ ($\mathcal{N}_\lambda(X)$ is the nerve of a finite closed covering obtained from a finite partitioning of X [B1]) and $|\mathcal{N}_\lambda(X)|_p = \varprojlim \{|\mathcal{N}_\lambda(X)|^p, \pi_{\lambda\mu}\} \subset |\mathcal{N}(X)|$ (K^p denotes the p -skeleton of the complex K), is an isomorphism.

In [B2] Berikashvili proposed new C_1 and C_2 axioms and the universal coefficient formula as one more new axiom as well:

C_1 : If a continuous map $f : X \rightarrow Y$ induces an isomorphism $f^* : \check{H}^n(Y; Z) \rightarrow \check{H}^n(X; Z)$ for $n < p$, then for $n < p - 1$ a homomorphism $f_* : \bar{H}_n(X; Z) \rightarrow \bar{H}_n(Y; Z)$ is an isomorphism as well.

C_2 : If a continuous map $f : X \rightarrow Y$ is surjective and $\bar{H}_n(f^{-1}(y), *) = 0$ for each $y \in Y$ and $n < p$, then for $n < p$ a homomorphism $f_* : \bar{H}_n(X; Z) \rightarrow \bar{H}_n(Y; Z)$ is an isomorphism.

D : For each pairs (X, A) there exists a functorial exact sequence

(1.8)

$$0 \rightarrow \text{Ext}(\check{H}^{n+1}(X, A); G) \rightarrow \bar{H}_n(X, A; G) \rightarrow \text{Hom}(\check{H}^n(X, A); G) \rightarrow 0,$$

where $G = \bar{H}_0(*)$.

Consequently, in [B2] the following is proved:

Theorem 1.5. (see theorem 4.4 [B2]) *The Steenrod-Sitnikov homology theory defined on the category Top_C^2 of compact Hausdorff pairs with coefficients any module G uniquely is characterized by the Eilenberg-Steenrod axioms with one of the following 4 systems of axioms: 1) A, B, C axioms; 2) axiom D ; 3) A, B, C_1 axioms; 4) A, B, C_2 axioms for the finite generated abelian group.*

In [IM] H. Inasaridze and L. Mdzinarishvili gave one more different axiomatic system using the modified form of the continuity axiom, as it is called, partial continuity axiom:

PC (partial continuity axiom): Let (X, A) be the inverse limit of inverse system $\{(X_\lambda, A_\lambda), p_{\lambda, \lambda'}, \Lambda\}$ of compact polyhedra, then for each integer n there is a functorial exact sequence

$$(1.9) \quad 0 \rightarrow \varprojlim^1 H_{n+1}(X_\lambda, A_\lambda) \rightarrow \bar{H}_n(X, A) \rightarrow \varprojlim H_n(X_\lambda, A_\lambda) \rightarrow 0.$$

Using the partial continuity axiom in the paper [M] L. Mdzinarishvili defined a nontrivial external extension \bar{H}_* of the homology theory H_* defined on the category \mathcal{Pol}^2 of compact polyhedra pairs to the category Top_C^2 of compact Hausdorff pairs [M]. In particular, homological sequence \bar{H}_* defined on the category Top_C^2 is called extension of homology theory H_* (which is unique up to an isomorphism) defined on the category \mathcal{Pol}^2 , if on the subcategory \mathcal{Pol}^2 it is equivalent to H_* [M]. The homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n \in \mathbb{Z}}$ defined on the category Top_C^2 is called a nontrivial external extension of the homology theory $H_* = \{H_n\}_{n \in \mathbb{Z}}$ defined on the category \mathcal{Pol}^2 , if the following conditions are fulfilled:

1_{NT}) \bar{H}_* is an extension of the homology theory H_* ;

2_{NT}) the exact sequence

$$(1.10) \quad 0 \rightarrow \varprojlim^1 H_{n+1}(X_\lambda, A_\lambda) \rightarrow \bar{H}_n(X, A) \rightarrow \varprojlim H_n(X_\lambda, A_\lambda) \rightarrow 0$$

holds for any object $(X, A) \in \text{Top}_C^2$, any inverse system $\{(X_\lambda, A_\lambda), p_{\lambda, \lambda'}, \Lambda\}$ of compact polyhedra such that $(X, A) = \varprojlim \{(X_\lambda, A_\lambda), p_{\lambda, \lambda'}, \Lambda\}$ and $n \in \mathbb{Z}$;

3_{NT}) The commutative diagram

$$(1.11) \quad \begin{array}{ccc} \varprojlim^1 \bar{H}_{n+1}(X_\lambda, A_\lambda) & \rightarrow & \bar{H}_n(X, A) \\ \downarrow \varprojlim^1 \tilde{f}_* & & \downarrow f_* \\ \varprojlim^1 \bar{H}_{n+1}(Y_\gamma, B_\gamma) & \rightarrow & \bar{H}_n(Y, B) \end{array},$$

holds for any continuous mapping $f : (X, A) \rightarrow (Y, B)$ from Top_C^2 , where $\tilde{f}_* : \{H_n(X_\lambda, A_\lambda), (p_{\lambda, \lambda'})_*, \Lambda\} \rightarrow \{H_n(Y_\gamma, B_\gamma), (q_{\gamma, \gamma'})_*, \Gamma\}$ is mapping of the inverse systems;

4_{NT}) \bar{H}_* satisfies the exactness axiom.

In [M] a homological sequence \bar{H}_* defined on the category Top_C^2 of compact Hausdorff pairs is called:

- 1) a homology theory in the Eilenberg-Steenrod sense if it satisfies the axioms of homotopy, excision, exactness and dimension;
- 2) a homology theory in the Milnor sense if it satisfies the axioms of homotopy, excision, exactness, dimension, relative homeomorphism and cluster axioms;
- 3) a homology theory in the Berikashvili sense if it satisfies the axioms of homotopy, excision, exactness, dimension and A, B and C axioms.

In [M] it is shown that any nontrivial external extension is homology theory in the Eilenberg-Steenrod, in the Milnor as well as in the Berikashvili sense:

Theorem 1.6. (see theorem 1.2 in [M]) *if \bar{H}_* is a nontrivial external extension of the homology theory H_* to the category Top_C^2 , then it is a theory in the Eilenberg-Steenrod sense.*

Theorem 1.7. (see theorem 1.3 in [M]) *if \bar{H}_* is a nontrivial external extension of the homology theory H_* , defined on the category Top_C^2 , then it is a homology theory in the Milnor sense.*

By theorem 1.2 and theorem 1.7 the following is true:

Corollary 1.8. *On the category Top_{CM}^2 of compact metric pairs the \bar{H}_* is a nontrivial external extension if and only if it is the homology theory in the Milnor sense.*

Theorem 1.9. (see theorem 1.5 in [M]) *if \bar{H}_* is a nontrivial external extension of the homology theory H_* defined on the category Top_C^2 , then it is a homology theory in the Berikashvili sense.*

Consequently, the following is obtained:

Corollary 1.10. *(see corollary 1.4 in [M]) Any nontrivial external extension of the homology theory H_* defined on the category Top_C^2 is isomorphic to the Steenrod homology theory.*

Note that there are many different constructions of an exact homology theory, but all of them are functors of the second argument as well: For each short exact sequence

$$(1.12) \quad 0 \rightarrow G \rightarrow G' \rightarrow G'' \rightarrow 0,$$

there is the functorial natural long exact sequence:

$$(1.13) \quad \dots \rightarrow \bar{H}_{n+1}(X; G'') \rightarrow \bar{H}_n(X; G) \rightarrow \bar{H}_n(X; G') \rightarrow \bar{H}_n(X; G'') \rightarrow \dots$$

Therefore, for a homology theory it is natural to consider it as a bifunctor. In the paper [I] H. Inasaridze described exact bifunctor homology theory using the continuity property for the infinitely divisible abelian groups. In particular, [I] the following is proved:

Theorem 1.11. *(see theorem 1 in [I]) There exists one and only one exact bifunctor homology theory on the category Top_C^2 of compact Hausdorff pairs with coefficients in the category of abelian groups (up to natural equivalence) which satisfies the axioms of homotopy, excision, dimension, and continuity for every infinitely divisible group.*

Therefore, for the Steenrod homology theory there are different axiomatic systems, but it is not known what the relation between them is and which one is the minimal one in the axiomatic sense. In the second part we will study this problem.

2 Relations between Different Axiomatic Systems

In this paper we will say that a homological sequence \bar{H}_* defined on the category Top_C^2 of compact Hausdorff pairs is:

- 1) a homology theory in the Berikashvili sense if it satisfies the axioms of homotopy, excision, exactness, dimension and axiom D (The Universal Coefficient Formula);
- 2) a homology theory in the Mdzinarishvili sense if it is a nontrivial external extension;

3) a homology theory in the Inasaridze sense if it is the exact functor of the second argument and satisfies the axioms of homotopy, excision, exactness, dimension and continuity for every infinitely divisible group.

Note that in the category \mathcal{Ab} of abelian groups G is infinitely divisible group if and only if it is injective. Therefore, in our paper instead of the term "infinitely divisible group" we will use "injective group".

Theorem 2.1. *If \bar{H}_* is a homology theory in the Inasaridze sense, defined on the category Top_C^2 of compact Hausdorff pairs, then it is the homology theory in the Berikashvili sense.*

Proof. For each compact Hausdorff space $X \in Top_C^2$ consider an inverse system $\mathbf{X} = \{X_\lambda, p_{\lambda, \lambda'}, \Lambda\}$ of compact polyhedra such that $X = \varprojlim \mathbf{X}$. By the condition of the theorem, for each injective group G we have an isomorphism:

$$(2.1) \quad \bar{H}_n(X; G) = \bar{H}_n(\varprojlim X_\lambda; G) \approx \varprojlim H_n(X_\lambda; G).$$

For each compact polyhedra X_λ we have the exact sequence [Mar]:

$$(2.2) \quad 0 \rightarrow Ext(H^{n+1}(X_\lambda); G) \rightarrow H_n(X_\lambda; G) \rightarrow Hom(H^n(X_\lambda); G) \rightarrow 0,$$

which induces the long exact sequence:

$$(2.3) \quad \begin{aligned} 0 \rightarrow \varprojlim Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow \varprojlim Hom(H^n(X_\lambda); G) \rightarrow \\ \varprojlim^1 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^1 H_n(X_\lambda; G) \rightarrow \varprojlim^1 Hom(H^n(X_\lambda); G) \rightarrow \\ \varprojlim^2 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^2 H_n(X_\lambda; G) \rightarrow \varprojlim^2 Hom(H^n(X_\lambda); G) \rightarrow \dots \end{aligned}$$

Note that for each injective group G the functor $Ext(-; G)$ is trivial and by (2.3) we obtain the isomorphism:

$$(2.4) \quad \varprojlim H_n(X_\lambda; G) \approx \varprojlim Hom(H^n(X_\lambda); G).$$

If we apply the isomorphism $\varprojlim Hom(H^n(X_\lambda); G) \approx Hom(\varinjlim H^n(X_\lambda); G)$, then by (2.4) we obtain:

$$(2.5) \quad \varprojlim H_n(X_\lambda; G) \approx Hom(\varinjlim H^n(X_\lambda); G) = Hom(\check{H}^n(X); G).$$

Therefore, by (2.1) if G is an injective then

$$(2.6) \quad \bar{H}_n(X; G) \approx \varprojlim H_n(X_\lambda; G) \approx Hom(\check{H}^n(X); G).$$

Now consider any abelian group G and corresponding injective resolution:

$$(2.7) \quad 0 \rightarrow G \rightarrow G' \rightarrow G'' \rightarrow 0.$$

Apply to the sequence (2.7) by the functor $Hom(\check{H}^n(X); -)$. The groups G' and G'' are injective and so we have:

$$(2.8) \quad 0 \rightarrow Hom(\check{H}^n(X); G) \rightarrow Hom(\check{H}^n(X); G') \rightarrow Hom(\check{H}^n(X); G'') \rightarrow Ext(\check{H}^n(X); G) \rightarrow 0.$$

Therefore, for each integer $n \in N$ we have

$$(2.9) \quad Ker(Hom(\check{H}^n(X) : G') \rightarrow Hom(\check{H}^n(X) : G'')) \approx Hom(\check{H}^n(X); G),$$

$$(2.10) \quad Coker(Hom(\check{H}^n(X) : G') \rightarrow Hom(\check{H}^n(X) : G'')) \approx Ext(\check{H}^n(X); G).$$

Now apply sequence (2.7) by homological bifunctor $\bar{H}_*(X; -)$, which gives the following long exact sequence:

$$(2.11) \quad \cdots \rightarrow \bar{H}_{n+1}(X; G') \rightarrow \bar{H}_{n+1}(X; G'') \rightarrow \bar{H}_n(X; G) \rightarrow \bar{H}_n(X; G') \rightarrow \bar{H}_n(X; G'') \rightarrow \dots$$

Therefore, for each $n \in N$ we obtain the following short exact sequence:

$$(2.12) \quad 0 \rightarrow Coker(\bar{H}_{n+1}(X; G') \rightarrow \bar{H}_{n+1}(X; G'')) \rightarrow \bar{H}_n(X; G) \rightarrow Ker(\bar{H}_n(X; G') \rightarrow \bar{H}_n(X; G'')) \rightarrow 0.$$

By (2.6), (2.9), (2.10) and (2.12) we finally obtain the following short exact sequence:

$$(2.13) \quad 0 \rightarrow Ext(\check{H}^{n+1}(X; G)) \rightarrow \bar{H}_n(X; G) \rightarrow Hom(\check{H}^n(X; G) \rightarrow 0.$$

□

Theorem 2.2. *If \bar{H}_* is a homology theory in the Berikashvili sense, defined on the category Top_C^2 of compact Hausdorff pairs, then it is a homology theory in the Mdzinarishvili sense.*

Proof. For each compact Hsurdorff space $X \in Top_C^2$ consider an inverse system $\mathbf{X} = \{X_\lambda, p_{\lambda, \lambda'}, \Lambda\}$ of compact polyhedra such that $X = \varprojlim \mathbf{X}$. For each $\lambda \in \Lambda$ and any abelian group G consider the following exact sequence [Mar]:

$$(2.14) \quad 0 \rightarrow Ext(H^{n+1}(X_\lambda); G) \rightarrow H_n(X_\lambda; G) \rightarrow Hom(H^n(X_\lambda); G) \rightarrow 0,$$

which induces the long exact sequence

$$\begin{aligned}
 0 \rightarrow \varprojlim Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow \varprojlim Hom(H^n(X_\lambda); G) \rightarrow \\
 \varprojlim^1 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^1 H_n(X_\lambda; G) \rightarrow \varprojlim^1 Hom(H^n(X_\lambda); G) \rightarrow \\
 (2.15) \quad \varprojlim^2 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^2 H_n(X_\lambda; G) \rightarrow \varprojlim^2 Hom(H^n(X_\lambda); G) \rightarrow \dots
 \end{aligned}$$

Note that for each $\lambda \in \Lambda$ the cohomology group $H^n(X_\lambda; G)$ is finitely generated [Mar] and so by Corollary 1.5 in [HM] we have:

$$(2.16) \quad \varprojlim^r Ext(H^n(X_\lambda); G) = 0, \quad r \geq 1.$$

Therefore, by (2.15) and (2.16) we obtain the exact sequence:

$$(2.17) \quad 0 \rightarrow \varprojlim Ext(H^{n+1}(X_\lambda); G) \rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow \varprojlim Hom(H^n(X_\lambda); G) \rightarrow 0.$$

Naturally, there exists the commutative diagram:

$$\begin{array}{ccccccc}
 (2.18) \quad 0 \rightarrow Ext(\check{H}^{n+1}(X); G) & \rightarrow & \bar{H}_n(X; G) & \rightarrow & Hom(\check{H}^n(X); G) & \rightarrow & 0 \\
 & & \downarrow \psi & & \downarrow \varphi & & \downarrow \chi \\
 0 \rightarrow \varprojlim Ext(H^{n+1}(X_\lambda); G) & \rightarrow & \varprojlim H_n(X_\lambda; G) & \rightarrow & \varprojlim Hom(H^n(X_\lambda); G) & \rightarrow & 0
 \end{array}$$

On the other hand, $\chi : Hom(\check{H}^n(X); G) = Hom(\varinjlim H^n(X_\lambda); G) \rightarrow \varprojlim Hom(H^n(X_\lambda); G)$ is an isomorphism and so

$$(2.19) \quad Ker\psi \approx Ker\varphi, \quad Coker\psi \approx Coker\varphi.$$

Therefore, we have the following commutative diagram of the exact sequences:

$$\begin{array}{ccccccc}
 (2.20) \quad 0 \rightarrow Ker\psi \rightarrow Ext(\check{H}^{n+1}(X); G) & \rightarrow & \varprojlim Ext(H^{n+1}(X_\lambda); G) & \rightarrow & Coker\psi & \rightarrow & 0 \\
 & \downarrow \approx & \downarrow & & \downarrow \approx & & \\
 0 \rightarrow Ker\varphi \rightarrow \bar{H}_n(X; G) & \rightarrow & \varprojlim H_n(X_\lambda; G) & \rightarrow & Coker\varphi & \rightarrow & 0
 \end{array}$$

As it is known [HM] by Proposition 1.2, for each direct system $\mathbf{A} = \{A_\alpha, \pi_{\alpha, \alpha'}, A\}$ of abelian groups there exists a natural exact sequence:

$$\begin{aligned}
 0 \rightarrow \varprojlim^1 Hom(A_\alpha; G) &\rightarrow Ext(\varinjlim A_\alpha; G) \rightarrow \varprojlim Ext(A_\alpha; G) \rightarrow \\
 (2.21) \quad &\rightarrow \varprojlim^2 Hom(A_\alpha; G) \rightarrow 0.
 \end{aligned}$$

Therefore, for the direct system $H^*(\mathbf{X}) = \{H^n(X_\lambda), \pi_{\lambda, \lambda'}, \Lambda\}$ of cohomological groups we have:

$$0 \rightarrow \varprojlim^1 Hom(H^{n+1}(X_\lambda); G) \rightarrow Ext(\varinjlim H^n(X_\lambda); G) \rightarrow \varprojlim Ext(H^n(X_\lambda); G) \rightarrow$$

$$(2.22) \quad \varprojlim^2 Hom(H^{n+1}(X_\lambda); G) \rightarrow 0$$

and consequently by (2.20) we obtain the exact sequence:

$$0 \rightarrow \varprojlim^1 Hom(H^{n+1}(X_\lambda); G) \rightarrow \bar{H}_n(X; G) \rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow$$

$$(2.23) \quad \varprojlim^2 Hom(H^{n+1}(X_\lambda); G) \rightarrow 0.$$

On the other hand for each $\lambda \in \Lambda$ the cohomology group $H^{n+1}(X_\lambda; G)$ is finitely generated [Mar] and by Corollary 1.5 (see 2e, 3e) we have:

$$(2.24) \quad \varprojlim^2 Hom(H^{n+1}(X_\lambda); G) = 0,$$

$$(2.25) \quad \varprojlim^1 Hom(H^{n+1}(X_\lambda); G) = \varprojlim^1 H_{n+1}(X_\lambda; G).$$

Therefore, by (2.23), (2.24) and (2.25) we obtain:

$$(2.26) \quad 0 \rightarrow \varprojlim^1 H_{n+1}(X_\lambda; G) \rightarrow \bar{H}_n(X; G) \rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow 0.$$

□

Theorem 2.3. *If a homology theory \bar{H}_* in the Mdzinarishvili sense, defined on the category Top_C^2 of compact Hausdorff pairs, is an exact functor of the second argument, then it is a homology theory in the Inasaridze sense.*

Proof. We should prove that \bar{H}_* is continuous for each injective group G . For each compact Hausdorff space $X \in Top_C^2$ consider an inverse system $\mathbf{X} = \{X_\lambda, p_{\lambda, \lambda'}, \Lambda\}$ of compact polyhedra such that $X = \varprojlim \mathbf{X}$. By the condition of the theorem, for each abelian group G we have the following short exact sequence:

$$(2.27) \quad 0 \rightarrow \varprojlim^1 H_{n+1}(X_\lambda; G) \rightarrow \bar{H}_n(X; G) \rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow 0.$$

Therefore, we should show that for each injective group G the first derivative is trivial:

$$(2.28) \quad \varprojlim^1 H_{n+1}(X_\lambda; G) = 0.$$

Indeed, for each compact polyhedra X_λ we have the sequence [Mar]:

$$(2.29) \quad 0 \rightarrow Ext(H^{n+1}(X_\lambda); G) \rightarrow H_n(X_\lambda; G) \rightarrow Hom(H^n(X_\lambda); G) \rightarrow 0,$$

which induces the long exact sequence:

$$\begin{aligned}
0 \rightarrow \varprojlim Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim H_n(X_\lambda; G) \rightarrow \varprojlim Hom(H^n(X_\lambda); G) \rightarrow \\
\varprojlim^1 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^1 H_n(X_\lambda; G) \rightarrow \varprojlim^1 Hom(H^n(X_\lambda); G) \rightarrow \\
(2.30) \quad \varprojlim^2 Ext(H^{n+1}(X_\lambda); G) &\rightarrow \varprojlim^2 H_n(X_\lambda; G) \rightarrow \varprojlim^2 Hom(H^n(X_\lambda); G) \rightarrow \dots
\end{aligned}$$

As it is known [Mar], for each $\lambda \in \Lambda$ the cohomology group $H^n(X_\lambda; G)$ is finitely generated and by Corollary 1.5 in [HM] we have:

$$(2.31) \quad \varprojlim^r Ext(H^{n+1}(X_\lambda); G) = 0, \quad r \geq 1.$$

Therefore, by (2.30) and (2.31) we obtain the isomorphism:

$$(2.32) \quad \varprojlim^1 H_n(X_\lambda; G) \approx \varprojlim^1 Hom(H^n(X_\lambda); G).$$

On the other hand, for each direct system $H^*(\mathbf{X}) = \{H^n(X_\lambda), \pi_{\lambda, \lambda'}, \Lambda\}$ of abelian groups we have:

$$0 \rightarrow \varprojlim^1 Hom(H^{n+1}(X_\lambda); G) \rightarrow Ext(\varinjlim H^n(X_\lambda); G) \rightarrow \varprojlim Ext(H^n(X_\lambda); G) \rightarrow$$

$$(2.33) \quad \varprojlim^2 Hom(H^{n+1}(X_\lambda); G) \rightarrow 0.$$

By (2.33) for each injective abelian group G we obtain:

$$(2.34) \quad \varprojlim^1 Hom(H^{n+1}(X_\lambda); G) = 0.$$

Therefore, by (2.32) and (2.33) we obtain (2.28). Therefore, for each injective group G we have

$$(2.35) \quad \bar{H}_n(X; G) \approx \varprojlim H_n(X_\lambda; G)$$

□

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